Lecture 8: Sequence, Series and Progression

8.1 Introduction

Sequences and series are sets of numbers that indicate a particular pattern of numbers. These patterns of numbers enable us to determine the next number from the previous ones. Therefore, in this lecture we shall discuss sequences and series.

Learning Objectives

At the end of the lecture, you should be able to:

i. Define a sequence and series.
ii. Explain the arithmetic progression.
iii. Describe the nth term of an arithmetic progression.
iv. Calculate the sum of n terms of an arithmetic progression.
v. Identify the nth term of a geometric progression.
vi. Calculate the sum of n term of a geometric progression.

8.2 Sigma notation

Sigma ($\sum$) is the 18th letter of the greek alphabet. It is applied in mathematics to indicate the sum of all values in range of series.

$$\sum x_n = x_1 + x_2 + \ldots + x_n$$

$$\sum_{i=1}^{n} \boldsymbol{x} = x_1 + x_2 + \ldots + x_n$$

Example:

Evaluate the indicated sum $\sum_{k=1}^{7} [\mathbf{1} + (-\mathbf{1})^{K}]$

Solution

$$\sum_{k=1}^{7}[\mathbf{2} + (-\mathbf{1})^{K}]$$

$= [\mathbf{2} + (-\mathbf{1})^{\mathbf{1}}] + [\mathbf{2} + (-\mathbf{1})^{\mathbf{2}}] + [\mathbf{2} + (-\mathbf{1})^{\mathbf{3}}] + [\mathbf{2} + (-\mathbf{1})^{\mathbf{4}}] + [\mathbf{2} + (-\mathbf{1})^{\mathbf{5}}] + [\mathbf{1} + (-\mathbf{1})^{\mathbf{6}}] + [\mathbf{2} + (-\mathbf{1})^{\mathbf{7}}]$

$$= (\mathbf{2} - \mathbf{1}) + (\mathbf{2} + \mathbf{1}) + (\mathbf{2} - \mathbf{1}) + (\mathbf{2} + \mathbf{1}) + (\mathbf{2} - \mathbf{1}) + (\mathbf{2} + \mathbf{1}) + (\mathbf{2} - \mathbf{1})$$

$$= \mathbf{1} + \mathbf{3} + \mathbf{1} + \mathbf{3} + \mathbf{1} + \mathbf{3} + \mathbf{1} = \mathbf{13}$$

The indicated sum = 13

8.3 Sequence of Numbers

There are sets of numbers, which shows particular patterns, some of these are sets of natural numbers 1, 2, 3,…, the set of even numbers 2, 4, 6,…, the set of odd numbers 1, 2, 5, 7,…, etc. For any of these patterns, it is simple to determine the next number from the previous ones if we know the order by which successive terms are found

What is a Sequence?

A sequence is an arrangement of a set of numbers in a definite order. For example, the pattern 1, 2, 4, 7, 11,… indicates that the first is 1, the second is 2, the third is 4, the fourth is 7 and the fifth is 11, etc. the pattern is such that the difference between consecutive numbers follows the pattern of natural numbers. Therefore, the sixth number that follows after 11 is 16.

Every numbers of a particulars pattern is called a term. Each term has a particular position value such as the first term, the second tern, and so on. Using such naming, a term corresponding to the last one is called the nth term.

In other words, we can say that a sequence or progression is a set of numbers or algebraic expressions which can be obtained from the proceeding one by a definite order. Each of the numbers or expressions forming the set, which is called a term of sequence. Thus the definition of the sequence requires:

(i) The first term;

(ii) The number of terms and,

(iii) The order by which successive terms can be found.

The following examples indicate the order which can be used to determine the n^{th} term in the set of sequences:

(i) 1, 3, 5, 7, ……………..$2n - 1$

(ii) 1, 4, 9, 16, ……………n^2

(iii) x, x^2, x^3, x^4……………x^n

The $2n - 1$, n^2 and x^n are the orders used to determine the n^{th} term of (i), (ii), (iii) respectively.

Example:

Find the fifth term of the sequence

1, 4, 7, 10,…

Solution:

The pattern shows that the difference between consecutive terms is 3. Therefore, every next term is determined by adding a 3 to the previous one. Thus, the fifth term (5^{th} term) $= 10 + 3 = 13$.

Example:

Given the sequence 2, 4, 6, 8, 10… find the:

(i) The first (1^{st}) term

(ii) The fourth (4^{th}) term

(iii) The nth term

Solution:

From the pattern of a set of numbers given, 2, 4, 6, 8, 10… the first term is 2 and the fourth term is 8 i.e.

The 1^{st} term is $2 = 2 \times 1$

The 2^{nd} term is $4 = 2 \times 2$

The 3^{rd} term is $6 = 2 \times 3$

The 4^{th} term is $8 = 2 \times 4$

The n^{th} term is $2 \text{ x } n = 2n$.

8.4 Properties of Sequence

In the sequence 5, 7, 9, 11, 13 … 2(n -1)…

Where, each term after the first is obtained by adding 2 to the preceding term, is an example of an arithmetic sequence. The sequence 5, 10, 20, 40, 80 … $5(2)^{n-1}$ where each term after the first is obtained by multiplying the preceding term by 2, is an example of a geometric sequence.

8.5 Series of Numbers

In the previous section we discussed about sequences, while in this section we shall deal with the concept of series and its mathematical calculation.

What is Series?

Series can be defined as an expression when the terms or the sequence are linked together with signs of addition or subtraction. The following examples show series.

(i) $1 + 3 + 5 + \ldots + 20 + 21$

(ii) $x - x^2 + x^3 \ldots$

(iii) $-2 -4 -6 -8 -10 \ldots.$

(i), (ii), and (iii) above are all series.

Finite and Infinite Series

If series end after a finite number of terms, then it is said to be a finite one. On the other hand, series is infinite if it does not have an end.

Example:

$1 + 3 + 5 + 7$ is a finite series.

$1 + 3 + 5 + 7 + \ldots$ is an infinite series

Classification of Series

Series are classified into two classes, called the Arithmetic and Geometric Series. Both of them are (Arithmatic and Geometric) are also known as progressions.

8.6 Arithmetic Progression (AP)

Arithmetic progression is the series in which each item is obtained from the proceeding one by adding or subtracting a certain constant value. That constant value is called common difference (d).

Example:

$1 + 2 + 3 + 4 + \ldots + 99$

$-1 - 3 - 5 - 7 - \ldots$

$x + 5x + 9x + 13x + \ldots$

These three sets of series are arithmetic progression with common differences 1, –2, and $4x$ respectively.

The n^th^ Term of an Arithmetic Progression

If n is the number of terms of an Arithmetic Progression (AP), then the nth term is denoted by A_n and the common difference by d.

The general formula for the nth term is given by:

n^{th} term $= (A_n) = a + (n-1)d$

$$\boldsymbol{A_n = a + (n-1)d}$$

Where,

$A_n =$ n^{th} term of an A. P

$a =$ First term of an A.P

$n =$ Number of terms

$d =$ Common difference

Example:

Consider the following Progression: a, a + d, a + 2d, a + 3d

1st term = $A_n = (A_1) = a$

2nd term = $A_n = (A_2) = a + d$

3rd term = $A_n = (A_3) = a + 2d$

4th term = $A_n = (A_4) = a + 3d$

Here the coefficient of d in any term is one less than the number of terms.

The nth term of Arithmetical Progression (A.P) is obtained by: $$A_n = a + (n-1)d$$

Example:

Consider the problem of counting in 2s starting with number 1, it implies that the first term (a) is 1 and the common difference (d) is 2. Find the 24th term of an arithmetic progression.

Solution:

(i) Given

$a = 1;\ \ d = 2;\ n = 24$

Using the formula for nth term, we obtain

$A_n = a + (n-1)d$

$A_{24} = 1 + (24 - 1)2$

$A_{24} = 1 + 23\ x\ 2 = 47$

Therefore the 24th term is 47

Example:

The first term an arithmetic progression is 6 and the common different is 5. Find

1. The third term

2. The nth term

Solution

(i) Given $a = 6$; $d = 5$; $n = 3$

using the formulae of A.P

$A_n = a + (n - 1)\ d$

$A_3 = 6 + (3 - 1)\ 5 = 6 + (2 \times 5) = 16$

$A_3 = 16$

(ii) nth term

$A_n = a + (n - 1)d$

$A_n = 6 + (n - 1)5$

$A_n = 6 + 5\mathrm{n} - 5$

$A_n = 5n + 1$

Sum of nth Terms of an Arithmetic Progression

Since the method of adding the terms of arithmetic progression is too long, mathematicians have developed the formula that can be used to determine the sum of nth terms of an arithmetic progression as given below:

Let S_n = the sum of the nth terms of an Arithmetic Progression.

Then,

$Sn = a + (a + d) + (a + 2d) + \ldots + (\boldsymbol{A_n} - 2d) + (\boldsymbol{A_n} - d) + \boldsymbol{A_n}$ ...(1)

Writing the terms in reverse order

$Sn = \boldsymbol{A_n} + (\boldsymbol{A_n} - d) + (\boldsymbol{A_n} - 2d) + \ldots + (a + 2d) + (a + d) + a$ ...(2)

Adding equations (1) and (2) together gives

$2Sn = (a + \boldsymbol{A_n}) + (a + \boldsymbol{A_n}) + (a + \boldsymbol{A_n}) + \cdots + (\&a + \boldsymbol{A_n}) + (\&a + \boldsymbol{A_n}) + (a + \boldsymbol{A_n})$

There are n terms in this series.

$$2Sn = n(a + \boldsymbol{A_n})$$

$$Sn = \frac{n}{2}(a + \boldsymbol{A_n})$$

but $\boldsymbol{A_n} = \boldsymbol{a} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d}$

$$Sn = \frac{n}{2}[a + \boldsymbol{a} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d}]$$

$$Sn = \frac{n}{2}[2\boldsymbol{a} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d}]$$

Sum of n terms of an arithmetic progression is determined by:

$$Sn = \frac{n}{2}[2\boldsymbol{a} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d}]$$

Example:

Find the sum of the first sixteen terms of the following arithmetic progression: 3+10+17+…

Solution:

$$a = 3,\ d = 7,\ n = 16$$

$$\begin{aligned} A_{16} &= \frac{n}{2}\ (2a + (n-1)\ d) \\ &= \frac{16}{2}\ [(2 \times 3) + (16 - 1)7] \\ &= 8\ [6 + (15 \times 7)] \\ &= 8\ (6 + 105) = 8\ (111) \\ A_{16} &= 888 \end{aligned}$$

Example:

The 4th term of an A.P is 22 and the 7th term is 40. Determine the first term, common difference, and hence the sum of the first 12 terms.

Solution:

All we need is:

(a) $\boldsymbol{A_n = a + (n - 1)d}$

(b) d = common difference

(c) $Sn = \frac{n}{2}[2\boldsymbol{a} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d}]$

Solve the simultaneous equation in order to obtain *a* and *d*, then find S_{12}

$\boldsymbol{A_4 = a + 3d = 22}$ ...(1)

$\boldsymbol{A_7 = a + 6d = 40}$ ...(2)

$\boldsymbol{d = 6}$ **and** $\boldsymbol{a = 4}$

$$\underline{\text{Sn} = 444}$$

Example:

Find three numbers in arithmetical progression such that their sum is 15 and their product is 45.

Solution:

Let $(a - d), a, (a + d)$ be the required numbers:

$$(a - d) + a + (a + d) = 15$$
$$3a = 15$$
$$a = 5$$
$$(a - d)\, a\, (a + d) = 45$$
$$(5 - d)\, 5\, (5 + d) = 45$$

Divide by 5

$$(5 - d)(5 + d) = 9$$
$$25 - d^2 = 9$$
$$d^2 = 16$$
$$d = \pm 4$$
$$\underline{\boldsymbol{d = 4, \quad -4}}$$

Example:

The sum of the first eight terms of A P is 60 and the sum of the next six terms (from the ninth to the fourteen) is 108. Find the first term and the common difference.

Solution:

$S_8 = 60$ ...(1)

$S_{14} = 60 + 108 = 168$ ...(2)

$$Sn = \frac{n}{2}[2\boldsymbol{a} + (\boldsymbol{n} - \mathbf{1})\boldsymbol{d}]$$

$$60 = \frac{8}{2}[2a + (8 - 1)d]$$

$$60 = 4(2a + 7d)$$

$$15 = (2a + 7d)$$

$2a + 7d = 15$ ...(3)

$$168 = \frac{14}{2}[2a + (14 - 1)d]$$

$$168 = 7(2a + 13d)$$

$$24 = (2a + 13d)$$

$2a + 13d = 24$ ...(4)

Solving equations (4) and (5) gives:

a $= 2.25$, d$= 1.5$

∴ The first term $a = 2.25$

The common difference = 1.5

The Arithmetic Mean (AM)

Arithmetic mean is the middle value when three consecutive terms or numbers are in arithmetic progression.

For example, *a, m*, and *b*

$$a = 1st\ term$$

$$m = 2nd\ term$$

$$b = 3rd\ term$$

d is the common difference which is obtained by:

$d = m - a = b - m$

$$m + m = b + a$$

$$2m = b + a$$

$$m = \frac{b + a}{2}$$

Arithmetic Mean is given by:

$$\text{AM} = m = \frac{b + a}{2}$$

m is called the arithmetic mean of the two number

Example

Calculate the arithmetic mean of 3 and 27.

Solution:

$a = 3, b = 27$

AM $= m = \frac{b + a}{2}$

AM $= \frac{3 + 27}{2} = \frac{30}{2} = 15$

8.7 Geometric Progression

Geometric progression is the one whose next term is obtained by multiplying (or dividing) with certain constant number to the previously term. The constant number is called common ratio, *r*. The common ratio, *r*, of a geometric progression can be obtained by dividing any term by its immediate predecessor. For example, 1+2+ 4 + 8 + 16 +… is a geometric series with first term, $a = 1$, and common ratio, $r = \frac{8}{4} = 2$

The general form of a geometric series is where a = first term, r = common ratio.

Example:

$1 + 2 + 4 + 8 + \dots$

$1 + {}^{1}/_{3} + {}^{1}/_{9} + {}^{1}/_{27} + \dots + {}^{1}/_{243}$

$^{16}/_{27} - ^{8}/_{9} + ^{4}/_{3} + \ldots + ^{27}/_{243}$

The three examples given above are geometric progressions with common rations, 2, $\frac{1}{3}$ and $\frac{-3}{2}$ respectively.

The *n* Term of *a* Geometric Progression

If *n* is the number of terms of a geometric progression, the n^{th} term is denoted by G_n and the common ratio by *r*. Consider the GP: $a, ar, ar^2, ar^3 \ldots$

The n^{th} term is obtained by:

$G_n = ar^{n-1}$

Where, $a = 1^{st}$ term

$r =$ Common ratio

$n = n^{th}$ term of a geometric progression

Therefore,

$G_1 = a$ 1^{st} term

$G_2 = ar^1$ 2^{nd} term

$G_3 = ar^2$ 3^{rd} term

$G_4 = ar^3$ 4^{th} term

$G_n = ar^{n-1} = n^{th}$ term

The nth term of a geometric progression is determined by:

$$Gn = ar^{n-1}$$

Example:

Write down the eighth term of each of the following geometric progressions.

$12 + 6 + 3 + \ldots$

$2 + 4 + 8 + \ldots$

Solution:

(i) Given, $12 + 6 + 3 + \ldots$

From the formula of geometric progression:

$G_n = a\,r^{n-1}$

$a = 1^{st}$ term $=12$

$r =$ Common ratio: $^6/_{12}$:$^3/_6 = ½$

$n =$ 8^{th} term?

Then,

$G_8 = 12(½)^{8-1} = 12(½)^7 = {}^{12}/_{128} = {}^3/_{32}$

$G_8 = {}^3/_{32}$

(ii) Given, $2 + 4 + 8 + \ldots$

$G_n = ar^{n-1}$

$a = 1^{st}$ term $= 2$

$r =$ Common ratio $= 4/2:8/4 = 2$

$n = 8$

Then,

$G_8 = 2(2)^{8-1} = 2(2)^7 = 256$

$G_8 = 256$

The Sum of n Term of a Geometric Progression

The sum of n terms in GP is denoted by S_n and the formula for the sum of S_n is derived from the following:

$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}$ ...(1)

Multiply equation (1) by r we get,

$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$ ...(2)

Subtract equation (1) from (2)

$$rS_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n - 1)$$

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

for $r \neq 0;\ r > 1$

Therefore, the sum of n^{th} term of GP is given by this expression:

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

Also the following formula is applicable when $r \neq 1$, and $r < 1$

$$S_n - rS_n = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$

Example:

If fourth term of GP is 9 and the sixth term is 81. Find

(i) Common ratio

(ii) The first term

(iii) Sum of the first five terms

Solution:

(i) The common ratio $= r$ and first term =a

the fourth term (G_4) $ar^3 = 9$ ...(1)

the sixth term(G_6) $ar^5 = 81$ ...(2)

divide (1) from (2)

$$r^2 = 9$$

$$r = \pm 3$$

(ii) To find the 1st term of the given problem above

$r = +3$	$r = -3$
$ar^3 = 9$	$ar^3 = 9$

$$a \times 3^3 = 9$$
$$27a = 9$$
$$a = \frac{1}{3}$$
$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$
$$S_5 = \frac{\frac{1}{3}(3^5 - 1)}{(3 - 1)}$$
$$= \frac{\frac{1}{3}(243 - 1)}{2}$$
$$= \frac{1}{6} \times 242$$
$$S_5 = 40\frac{1}{3}$$

$$a \times (-3)^3 = 9$$
$$-27a = 9$$
$$a = -\frac{1}{3}$$
$$S_n = \frac{a(1 - r^n)}{(1 - r)}$$
$$S_5 = \frac{\frac{1}{3}[1 - (-3)^5]}{1 - (-3)}$$
$$= \frac{\frac{1}{3}[1 - (-243)]}{4}$$
$$= \frac{1}{12} \times 244$$
$$S_5 = 20\frac{1}{3}$$

Example:

Given that the first term of a geometric progression is 2 and its common ration is ½; find:

(i) The 7th term of the GP

(ii) The sum of the first five terms of the GP solution.

Solution:

Given, $a = 2$, $r = ½$

(i) $G_n = ar^{n-1}$

$$G_7 = ar^{7-1}$$
$$= ar^6$$
$$= 2 \times \left(\frac{1}{2}\right)^6$$
$$= \frac{1}{32}$$

(ii) $S_n = \frac{a(1-r^n)}{(1-r)}$ $\quad$ $r < 1$

$$S_5 = \frac{2\left[1 - \left(\frac{1}{2}\right)^5\right]}{1 - \frac{1}{2}}$$

$$= \frac{2\left(1 - \frac{1}{32}\right)}{\frac{1}{2}}$$

$$= 4\left(1 - \frac{1}{32}\right)$$

$$= 4 \times \frac{31}{32}$$

$$= \frac{31}{8}$$

$$= 3\frac{7}{8}$$

$$S_5 = 3\frac{7}{8}$$

Example 3

If 7, P, q and 189 are in geometric progression. Find the value of p and q.

Solution

Let

a = 1st term; n = n^{th} term of G.P; r = Common ratio; G_n = n^{th} term

Given,

$G_4 = 189$; $a = 7$; $r = 2$

$$G_4 = ar^{4-1}$$

$$189 = 7r^{4-1}$$

$$189 = 7r^3$$

$$r^3 = \frac{189}{7}$$

$$r^3 = 27$$

$$r = \sqrt[3]{27} = 3$$

$$p = G_2 = ar^{2-1} = 7(3^{2-1}) = 7 \times 3 = 21$$

$$q = G_3 = ar^{3-1} = 7(3^{3-1}) = 7 \times 3^2 = 63$$

Therefore, $p = 21$ and $q = 63$

Geometric Mean

Geometric mean is the square root of the product of values in the given set of numbers. For example, if a variate takes n values $x_1, x_2, ..., x_n$, then the geometric mean is the nth root of the product of these values, that is

$$\text{Geometric mean} = \sqrt{x_{1,} x_2 ... x_n}$$

Like the arithmetic mean, the geometric mean utilises all the information available. It can be shown that the geometric mean is always less than the arithmetic mean.

Example:

Find the geometric mean of 4 and 16.

Solution:

From the formula

Geometric mean = $\sqrt{x_{1,} x_2 ... x_n}$

Geometric mean $= \sqrt{4 \times 16} = \sqrt{64} = 8$

The same data, when the geometric mean is equal to 8, the arithmetic mean is equal to 10. That is;

$\text{AM} = \frac{4 + 16}{2} = \frac{20}{2} = 10$

Activity

Take this chess board and place 1 grain on the first square, 2 grains on second, 4 grains on third, 8 grains on fourth and so on....doubling each time till 64th square. Now answer…..

1. How many grains are on the 64^{th} square?

2. How many total grains are there on the chessboard?

Summary

A sequence is a special kind of function whose domain is the positive integers. The range of a sequence is the collection of terms that make up the sequence. Just as the word sequence implies, the order of the terms in a sequence is important. The first term of a sequence, for example, is found by taking the value of the function at 1; the second term is the value of the function at 2, and so on. Consider the sequence f (x) = x . The terms of the sequence, denoted a 1, a 2, a 3,…, a n are 1, 2, 3,…, n . Two important categories of sequences are arithmetic sequences, and geometric sequences. When the terms of a sequence are summed, the result is called a series. Some series increase without bound as n increases, but others approach a limit.

Exercise

1. Find the n^{th} term of the sequence: 1, 3, 5, 7…

2. Find the n^{th} term of the sequence: 3, 6, 9, 12,

3. The n^{th} term of a sequence is 4_{n-1}. Write down the first five terms.

4. The n^{th} term of a sequence is given by 2_{n+1}. Write down the tenth term.

5. Find the nth term of an arithmetic progression whose first term is $x + 2$ and the common difference is 3.

6. Write down the first 5 terms of the arithmetic progression and find the formula for the nth term.

7. The fourth term of AP is 15 and the seven term is 27 find the first term, common difference and twentieth term of the AP.

8. First term of an arithmetic progression is −12 and the last term is 40. If the sum of the terms is 196, find the number of terms and the common difference.

9. The first term of an arithmetic progression is 13 and the fifth term is 21. Find the common difference and the sum of the first ten terms.

10. Find the fourth term of the following geometric progression: $-4, 1, -\frac{1}{4}, \ldots$

11. The nth term of geometric progression series 4, 8, 16,… is 1024. Find n^{th}.

12. If the fourth term of a geometric progression is 9 and the sixth term is 81, find
    (a) The common ratio.
    (b) The first term

13. What is the sum of ten terms of the GP 2 – 6 +18 – 54 +…

14. The numbers n – 2, n, n +3 are consecutive terms of GP. Find n and the term after $n +$ 3.

15. The first three terms of geometric progression.

16. If a geometric progression is given by $G_n = 2n$. Find the sum of the first five terms.

17. Find three numbers in geometrical progression such that their sum is 26 and their product is 216.

18. Given that the first term of a geometric progression is 2 and its common ration is ½; find:
    (a) The 7th term of the G.P;
    (b) The sum of the first five terms of the GP solution.

19. Find the geometric mean of the following:

(a) 1, 4

(b) 1,2,4

(c) 10, 20, 30, 40, 50

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